3.5: Nonhomogeneous Equations and Undetermined Coefficients

In this section we consider general nonhomogeneous linear equations with constant coefficients; i.e. equations of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 = f(x). \tag{1}$$

We have already discussed the fact that if we find a particular solution for (1) y_p then

$$y = y_c + y_p$$

is a also a solution for where y_c is the complementary solution; i.e. the solution to the associated homogeneous equation. Thus our only task to find a general solution is to find y_p.

The **method of undetermined coefficients** is a straightforward way of doing this when $f(x)$ is sufficiently simple.

Example 1. Find a particular solution of $y'' + 3y' + 4y = 3x + 2$.

$y_p = Ax + B$

$y_p' = A$

$y_p'' = 0$

$0 + 3(A) + 4(Ax + B) = 3x + 2$

$A = \frac{3}{4},\ B = -1$

$y_p = \frac{3}{4}x - 1$

Example 2. Find a particular solution of $y'' - 47y = 2e^{3x}$.

$y_p = Ae^{3x}$

$y_p' = 3Ae^{3x}$

$y_p'' = 9Ae^{3x}$

$y_p'' - 47y_p = 9Ae^{3x} - 47Ae^{3x} = 2e^{3x}$

$A = \frac{-1}{17} \Rightarrow y_p = \frac{-1}{17}e^{3x}$

Example 3. Find a particular solution of $3y'' + y' - 2y = 2\cos x$.

$y_p = A\cos x + B\sin x$

$y_p' = -A\sin x + B\cos x$

$y_p'' = -A\cos x - B\sin x$

$2\cos x = 3y_p'' + y_p' - 2y = 3(A\cos x + B\sin x) + (-A\sin x + B\cos x) - 2(-A\cos x - B\sin x)$

$= (5A + B)\cos x + (5B - A)\sin x$

$\left.\begin{array}{l} 5B - A = 0 \\ 5B + A = 2 \end{array}\right\} \begin{array}{l} A = 1 \\ B = 1/5 \end{array} \Rightarrow y_p = \cos x + \frac{1}{5}\sin x.$

The idea should now be clear. Whatever different types of functions appear in $f(x)$; i.e. polynomial, exponential, or $\cos x$ and $\sin x$, you create the most general set of linearly independent functions that could have these as derivatives and use that as a trial function.

Example 4. Solve the initial value problem

$$y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x$$
$$y(0) = 1, \quad y'(0) = 2.$$

$r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0$

$Y_c = C_1 e^{2x} + C_2 e^{x}$

$Y_p = Ae^{-x} + B\cos 3x + C\sin 3x$

$Y_p' = -Ae^{-x} - 3B\sin 3x + 3C\cos 3x$

$Y_p'' = Ae^{-x} - 9B\cos 3x - 9C\sin 3x$

$Y_p'' - 3Y_p' + 2Y_p = 6Ae^{-x} + (-7B - 9C)\cos 3x + (9B - 7C)\sin 3x$

$A = \frac{1}{2}$, $\left.\begin{array}{l} -7B - 9C = -10 \\ 9B - 7C = 0 \end{array}\right\}$ $B = \frac{7}{13}$, $C = \frac{9}{13}$

$Y_p = \frac{1}{2}e^{-x} + \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x$ $\quad C_1 = \frac{6}{13}, C_2 = \frac{-1}{2}$

$Y = Y_p + Y_c = \frac{6}{13}e^{2x} - \frac{1}{2}e^{x} + \frac{1}{2}e^{-x} + \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x.$

Example 5. Find the general form of the particular solution of

$$y^{(3)} + 9y' = x\sin x + x^2 e^x.$$

$r^3 + 9r = 0$

$r(r^2 + 9) = 0$

$r = 0, \pm 3i$

$Y_c = C_1 + C_2\cos 3x + C_3\sin 3x$

Complementary Sol'n

Particular Solution

$Y_p = A\cos x + B\sin x + Cx\cos x + Dx\sin x + Ee^{2x} + Fxe^{2x} + Gx^2e^{2x}$

Now we know how to solve equations of the form (1) when the function $f(x)$ is sufficiently simple. Even more, we know how to find the most general solution by simply adding the particular and complementary solutions together. However, there is one more issue to tackle. What if the particular solution and the complementary solution are not linearly independent? Then we cannot hope to simply add them together because any of the coefficients in the complementary solution which are linearly dependent with the particular solution will necessarily be zero. Thus we cannot have the most general solution. Then we simply multiply y_p by an appropriate factor of x^s.

Example 6. Find a particular solution of $y^{(3)} + y'' = 3e^x + 4x^2$.

$r^3 + r^2 = 0$

$r^2(r+1) = 0$

$r = 0, 0, -1$

$Y_c = C_1 e^{-x} + C_2 + C_3 X$

$Y_p = Ae^x + Bx^2 + Cx^3 + Dx^4$

$A = \frac{3}{2},\ B = 4,\ C = -\frac{4}{3},\ D = \frac{1}{3}$

$Y_p = \frac{3}{2} e^x + 4x^2 - \frac{4}{3}x^3 + \frac{1}{3}x^4$

Example 7. Determine the appropriate form for a particular solution of

$$y'' + 6y' + 13y = e^{-3x}\cos 2x.$$

$r^2 + 6r + 13 = 0$

$(r+3)^2 + 4 = 0$

$y_c = e^{3x}(A\cos 2x + B\sin x)$

$y_p = xe^{3x}(C\cos 2x + D\sin 2x)$

Theorem 1. (Variation of Parameters)
If the nonhomogeneous equation $y'' + P(x)y' + Q(x) = f(x)$ has complementary function $y_c(x) = c_1y_1(x) + c_2y_2(x)$, then a particular solution is given by

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx,$$

where $W = W(y_1, y_2)$ is the Wronskian of the two independent solutions y_1 and y_2.

Example 8. Find a particular solution of the equation $y'' + y = \tan x$.

$\cancel{r^2 + r = 0}$

$\cancel{r(r+1) = 0}$

$\cancel{r = 0, -1}$

$\cancel{y_c = A + Be^{-x}}$

$\cancel{W(x) = \det\begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{vmatrix} = -e^{-x} - e^{-x} = -2e^{-x}}$

$\cancel{y_p = -1\int \frac{e^{-x}\tan x}{-2e^{-x}}dx + e^{-x}\int \frac{\tan x}{-2e^{-x}}dx}$

$r^2 + 1 = 0$

$y_c = A\cos x + B\sin x$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$y_p = -\cos x \int \frac{\sin x \tan x}{1}dx$

$+ \sin x \int \frac{\cos x \tan x}{1}dx$

$\cancel{= -\cos x \cdot -\cos x + \sin x \tan x}$

$= -\cos x \cdot (\sin x - \ln|\sec x + \tan x|)$

$+ \sin x \cdot -\cos x$

$= -\cos x \ln|\sec x - \tan x|$

Homework. 1-27, 31-37, 57-59 (odd)